Rhythmic hopping in a one-dimensional crisis map

Chihiro Seko and Kazuo Takatsuka
Graduate School of Human Informatics, Nagoya University, Nagoya 464-01, Japan

(Received 10 July 1995; revised manuscript received 11 January 1996)

It is shown that a time-evolving strange attractor formed by a dynamical contact of two logistic maps, a special case of the so-called interior crisis, can generate rhythmic hopping with a chaotic fluctuation between two domains on which the individual logistic maps are defined. A variety of distributions of the hopping time have been obtained with different choices of an evolution parameter in the map. The physical mechanism and the conditions for the rhythmic hopping are discussed. [S1063-651X(96)00107-9]

PACS number(s): 05.45.+b, 02.50.Wp, 05.40.+j, 87.10.+e

It is widely argued in the literature of chaos [1] that some rhythmic phenomena in nature, such as various physiological rhythms like the heart beat [2–4], can be correlated with fluctuation. The essential question is then whether or not this fluctuation might be chaos rather than random noise. Note that chaos implies deterministic randomness and complexity and is a concept counter to that of regularity and periodicity [1]. We show in the present report an example of a very simple chaotic element that can generate a rhythmic motion.

Periodic or quasiperiodic motions have been studied extensively in nonlinear dynamics [5] using the notions of limit cycle, torus, resonance, frequency locking, and so on. A number of simple differential equations are known that have limit cycle solutions to simulate neural oscillation [6]. In some chaos, strong periodic components can be found in a power spectrum in which almost discrete line spectrums are embedded in a ''continuous'' band (e.g., the Rössler attractor [7] and in the Hénon-Heiles system [8]). A periodic system on the other hand can become chaotic through a sequence of frequency locking in response to delivery of a periodic perturbation [9–11]. Conversely, as a bifurcation parameter is changed, chaos can be replaced by periodicity, resulting in periodic windows appearing in the bifurcation diagram [1,12].

Here we study a rhythmic hopping between two disjoint domains on which our one-dimensional map is defined. There are several examples in the literature [13–20] that have investigated the statistical properties of hopping or the related phenomenon of ''escape'' or ''extinction'' [14,20]. However, none of the prior works studied circumstances in which the motion had a strong periodic component (''rhythm'').

Our system, probably the simplest chaotic attractor that can indeed generate rhythm, is constructed as follows. Take a domain [0,2] in a one-dimensional space, two subdomains of which are denoted as $L=[0,1]$ and $R=[1,2]$. Let us consider a map $x_{n+1}=F(x_n)$ consisting of two logistic maps [21] such that

$$F(x)=\begin{cases} ax(1-x), & 0\leqslant x\leqslant 1\\ a(x-1)(x-2)+2, & 1\leqslant x\leqslant 2. \end{cases} \tag{1}$$

See Fig. 1. Similar one-dimensional maps have been studied previously [16,18,19]. When $a=4.0$, a trajectory $\{x_n|n=0,1,\dots\}$ starting from each domain remains there and undergoes strongly chaotic motion. If $a>4$, on the other hand, the two otherwise disjoint regions are brought into contact with one another, and a trajectory can wander over the entire range in a chaotic manner. This is merely a special case of the so-called interior crisis, first proposed by Grebogi, Ott, and Yorke [22–24].

Here a feature of time evolution is introduced into the above map by setting $a(t)=4.0+t^d$, where t is a discretized time measured in a unit time Δt (actually $\Delta t=0.01$). It is in the pioneering work of Lasota and Mackey [14] that an evolving map was considered for the first time. They studied an evolving logistic map having a slowly ''time dependent'' parameter such as $a(t)=\tilde{r}+\epsilon' t$ that is defined only in the domain $L=[0,1]$, from which a trajectory escapes (with no return) eventually after the value of the parameter a exceeds 4.0. This phenomenon was called extinction [14,20] and a beautiful analysis on the survival probability of a trajectory in $L=[0,1]$ was performed. In contrast to the case of extinc-

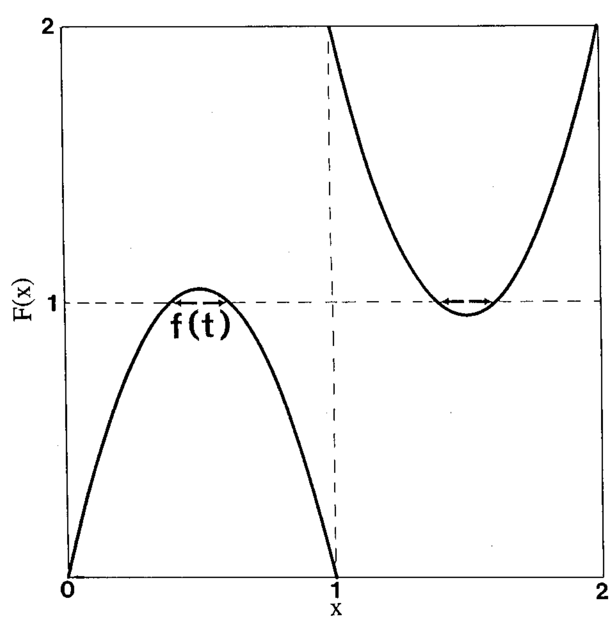

FIG. 1. A map causing an interior crisis. Unlike the ordinary (fixed) crisis, the diameter of the escaping region grows as a function of time.

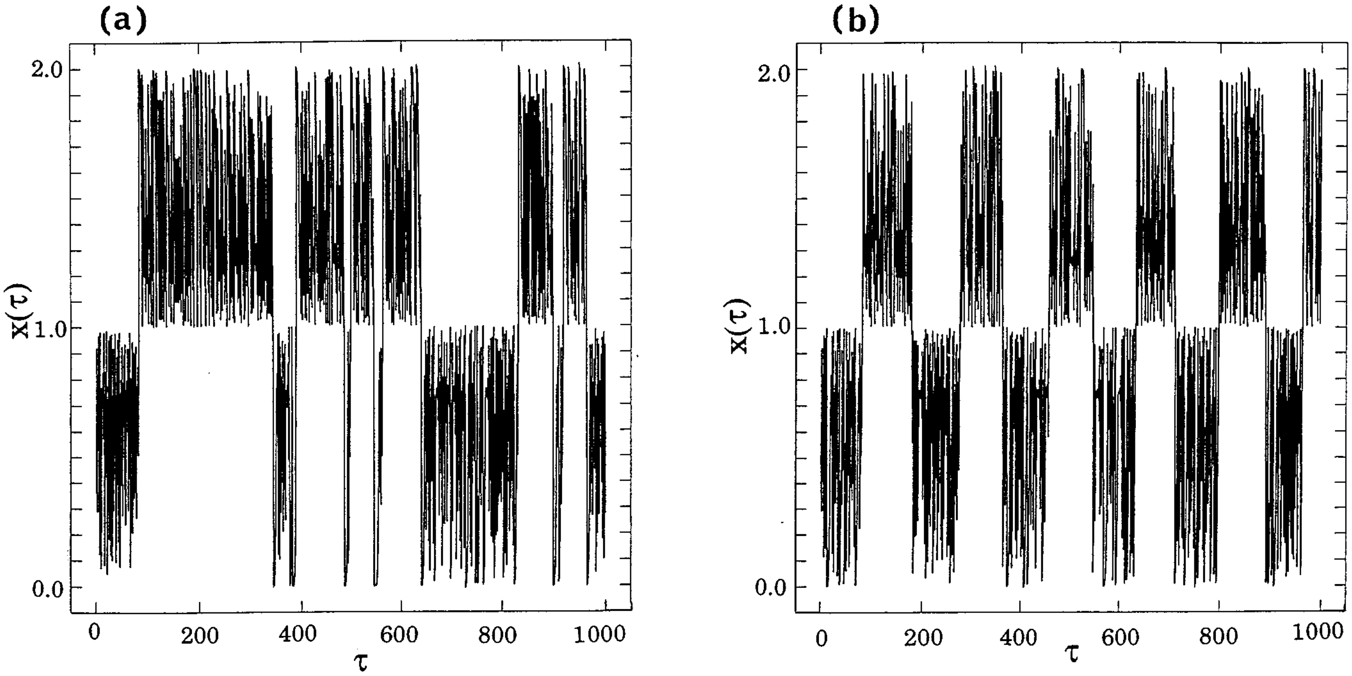

FIG. 2. The trajectories versus time. (a) A fixed crisis with the parameter $a=4.001$. (b) A breathing crisis of $a(t)=4.0+t^d$ with $d=20.0$.

tion, our map brings a trajectory back from $R=[1,2]$ to $L=[0,1]$, and vice versa. For example, let a trajectory start from an arbitrary point in, say, the L domain with $a=4.0$. As soon as it begins to move, we set $a(t)=4.0+t^d$. The positive exponent d is fixed. Since the gate diameter $f(t)$ through which the trajectory can escape from domain L grows larger (see Fig. 1), the trajectory eventually jumps into the neighboring subdomain after some finite lifetime. As soon as the trajectory gets into the neighboring domain, everything is reset; $t=0$ (the usual accumulated time is denoted by τ) and $a(0)=4.0$ (the gate is closed). Then, the trajectory resumes its course and the gate begins to open as before. In this way, the present system generates a behavior of the so-called renewal process. We now coarse grain the time series $\{x_n|n=0,1,\ldots\}$ by assigning the values to L or R depending on the domain where the trajectory stays, that is $LRLRLR\ldots$. The position of the trajectory in each piece of L or R changes in an approximately random manner. Surprisingly, however, it can make a rhythmic hopping between the two domains in spite of the fact that no source of periodic machinery has been installed.

Figure 2 shows two trajectories $x(\tau)$ ($\tau=n\Delta t$), for which (a) a is fixed at 4.001 and (b) $a(t)=4.0+t^d$ with $d=20$. We distinguish these two cases by terming the former and the latter, respectively, a fixed crisis and a breathing crisis. The points $\{x_n|n=0,1,...\}$ have been connected with straight lines in the figure. The initial conditions x_0 are randomly selected. A very clear distinction is noticed immediately in the two panels: The fixed crisis displays a chaotic motion, whereas the breathing crisis, though still somewhat chaotic, is much more regular. We shall thus call it "soft rhythm." These characteristics are quite generic, although the measure of the exceptional trajectories have not been determined.

The lifetime T is defined as the length of time that a trajectory remains in the subdomain L or R. Sampling an arbitrarily selected trajectory like the one shown in Fig. 2(b), one can count a distribution (or frequency) $P(T)$ of the lifetime. Figure 3 shows three different classes of $P(T)$; (a) a fixed crisis with $a=4.005$, (b) a breathing crisis with $d=2.0$, and (c) a breathing crisis with $d=20.0$. These distribution functions reflect the statistical nature behind the dynamics, and are identified as, respectively, (a) Poisson [14,19,20,26], (b) Wigner, and (c) Brody distributions [1,25]. Later we will show mathematically that this assignment is indeed the case. As will be deduced from the figure, the distribution becomes sharper as the evolution parameter d gets larger. For the choices of $a(t)$ shown in Fig. 3, it follows that a longer mean lifetime is associated with less fluctuation in $P(T)$, that is, more like a periodic motion. In particular, the longer side of the tails in the distributions becomes less marked for the larger d system.

Figure 4(a) shows the power spectrum of the time series of $LRLRLR\ldots$ for $d=2.0$, and 4(b) for $d=20.0$. It is quite clear that the breathing crisis with $d=20.0$ is sharply peaked, while for $d=2.0$ the spectrum is much more broadband. These spectra clearly indicate that the present *rhythm* is still *chaos*. The second and third peaks observed in Fig. 4(b) correspond to the third and fifth harmonics, respectively, of the fundamental frequency. They arise because the coarse-grained time series $LRLRLR\ldots$ forms a "rectangular" wave [see Fig. 2(b)].

The physical meaning of the present system is very clear. A deterministic diffusion process is materialized in the logistic map for $a\geqslant 4.0$, before the hopping takes place. Furthermore, the invariant density of the logistic map for $a=4.0$ is proportional to $[x(1-x)]^{-1/2}$ in [0,1.0] (see Ref. [1(a)]), which implies that a trajectory visits most of the ranges including the gate area with similar chances. On the other hand, the function $a(t)=4.0+t^d$ makes the channel open so that the diameter $f(t)$ is $f(t)=t^{d/2}(4+t^d)^{-1/2}\approx t^{d/2}/2$. Thus the chance for a trajectory to pass through the gate is typified

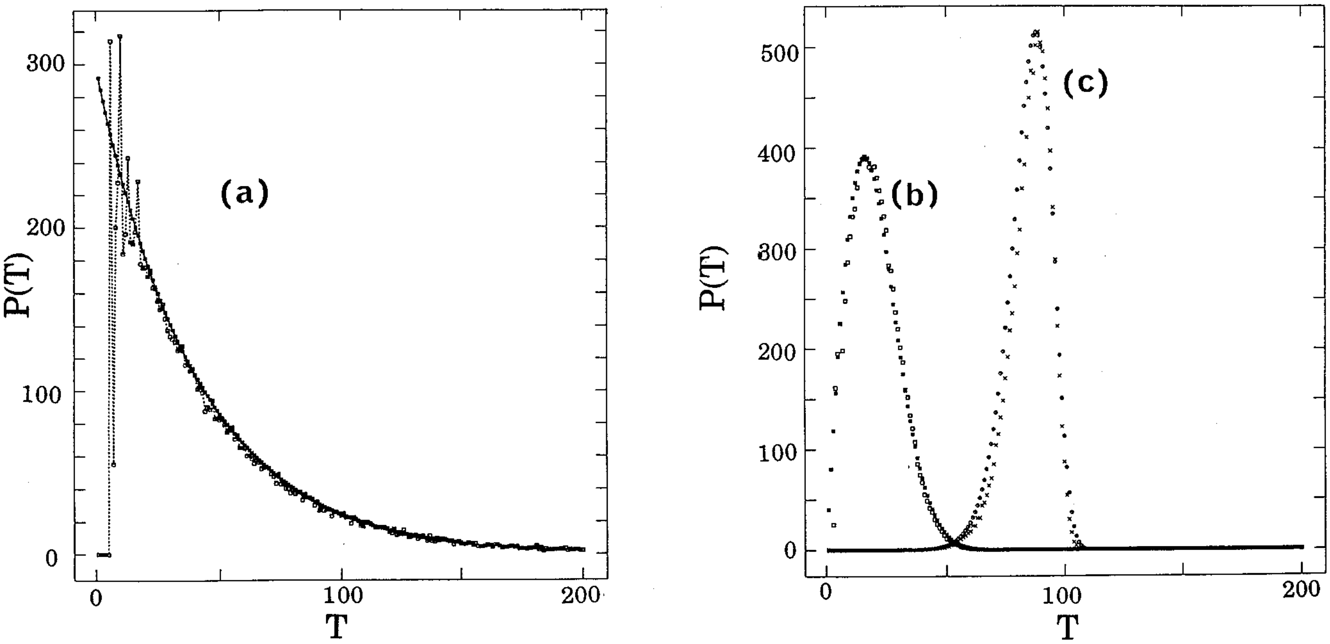

FIG. 3. The distributions of the lifetimes for both the domains. (a) A fixed crisis of $a=4.005$. The exponential curve represents the theory (Poisson distribution), while the ''experimental'' values, which are connected with a dotted line, deviate from the theory in the short time region. (b) A breathing crisis of $a(t)=4.0+t^d$ with $d=2.0$. The Wigner distribution expected from the theory (black squares) is so close to the numerically obtained values (white squares) that they are almost indistinguishable. (c) A breathing crisis with $d=20.0$. The experimental values (crosses) are well represented in terms of the Brody distribution (circles).

by this single evolution parameter d into three cases: the gate diameter $f(t)$ is (a) constant in time (for $d=0.0$, Poisson), (b) proportional to the time (for $d=2.0$, Wigner), and (c) accelerated as the time passes (for $d>2.0$, Brody). It turns out that only the third case leads to the soft rhythm. In other words, the rhythm can beat only under a condition that the gate begins to open very slowly in the early stage and later the diameter becomes wider quickly. In fact, it has been confirmed numerically that $f(t)$ which is not exactly of the form $f(t)=t^{d/2}$ can lead to soft rhythm as well, provided that the above condition is satisfied. The present diffusion-gating mechanism reminds us of single ion-channel recordings [2,3,26,27]. It would be necessary, however, to combine or embed these rhythmic elements in a large network system in order to simulate a complicated macroscopic system.

Next we consider how the distributions of $P(T)$ in Figs. 3(a)–3(c) can be understood on the basis of the above physical mechanism. Since the chaotic nature of the logistic map

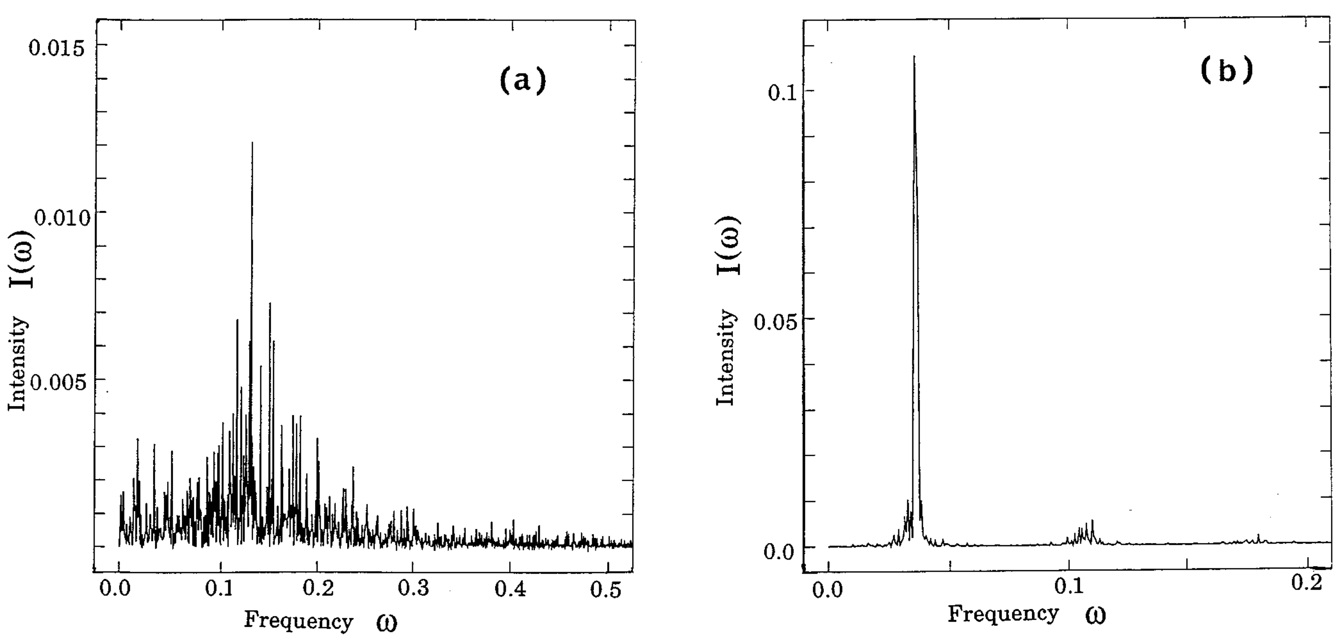

FIG. 4. Power spectra arising from the rhythmic motions $LRLRLR\ldots$. (a) A breathing crisis with $d=2.0$. (b) The same as in (a) but with $d=20.0$.

ensures that a trajectory visits any place with similar chances [1], $P(T)$ is expected to be governed only by the rate of increase of the diameter of the gate. If this is not the case, one should take account of the anisotropic distribution of the invariant density of a map under study. Let $f(t)\Delta t$ be a (unnormalized) probability for a trajectory to undergo a hopping in the time interval $[t,t+\Delta t]$, since the probability of hopping should be proportional to the diameter of the gate $f(t)$. Also, let $g(t)\Delta t$ be the probability for which hopping does not take place until time t (assuming the preceding hopping took place at $t=0$) and does in $[t,t+\Delta t]$. Then it follows that:

$$g(t)\Delta t=[1-f(0)\Delta t][1-f(\Delta t)\Delta t]\times[1-f(2\Delta t)\Delta t]\cdots\{1-f[(N-1)\Delta t]\Delta t\}\times f(t)\Delta t, \tag{2}$$

with $\Delta t=t/N$ and that

$$\ln g(t)=\lim_{N\to\infty}\sum_{j=0}^{N-1}\ln[1-f(j\Delta t)\Delta t]+\ln f(t) \cong -\int_0^t ds\ f(s)+\ln f(t), \tag{3}$$

where $f(j\Delta t)\Delta t \ll 1$ has been used. Equation (3) is rewritten as

$$g(t)=f(t)\exp\left(-\int_0^t ds\ f(s)\right). \tag{4}$$

Inserting $f(t)\approx t^{d/2}/2$, we finally have $g(t)\propto t^{d/2}\exp[-t^{(d/2)+1}/(d+2)]$. Thus $d=0$ [Fig. 3(a)], $d=2$ [Fig. 3(b)] and $d=20$ [Fig. 3(c)] give, respectively, the Poisson, Wigner, and Brody distributions. As can be confirmed in these figures, the numerically obtained distributions lie very close to the theoretical curves.

We have constructed a simple element of chaos that can generate a soft rhythm by allowing two logistic maps to come into contact dynamically, which we have called the breathing crisis. The mechanism and conditions giving rise to the soft rhythm have been discussed. The present study suggests that the dynamical contact of the arbitrary strange sets (attractors) can generate various types of rhythm. Although the focus of the present research has been placed on an aspect of chaotic dynamics, particularly on the new property of the interior crisis, the present system might lay a basis to interpret many renewal processes such as earthquake and rhythmic events in physical and biological sciences.

[1] As review articles of chaos, see, for instance, (a) A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer, Berlin, 1992); (b) R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, *Nonlinear Physics* (Harwood, Academic Chur, Switzerland, 1988); (c) M. Tabor, *Chaos and Integrability in Nonlinear Dynamics* (Wiley, New York, 1989); (d) A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise* (Springer, Berlin, 1994). See also Refs. [5,12,25].

[2] L. Glass and M. C. Mackay, *From Clocks to Chaos* (Princeton University Press, Princeton, New Jersey, 1988).

[3] (a)A. T. Winfree, *The Geometry of Biological Time* (Springer, Berlin, 1980); (b) *Chaos in Biological Systems*, edited by H. Degn, A. V. Holden, and L. F. Olsen (Plenum, New York, 1987); (c) *Rhythms in Physiological Systems*, edited by H. Haken and H. P. Koepchen (Springer, Berlin, 1991).

[4] B. J. West, *Fractal Physiology and Chaos in Medicine* (World Scientific, Singapore, 1990).

[5] (a) J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, Berlin, 1983); (b) T. Matsumoto, M. Komuro, H. Kokubu, and R. Tokunaga, *Bifurcations* (Springer, Berlin, 1993).

[6] A. L. Hodgkin and A. F. Huxley, J. Physiol. (London) **116**, 449 (1952); **116**, 473 (1952); **116**, 497 (1952); **117**, 500 (1952).

[7] D. Farmer, J. Crutchfield, H. Froeling, N. Packard, and R. Shaw, Ann. N.Y. Acad. Sci. **357**, 453 (1980).

[8] K. Takatsuka, Bull. Chem. Soc. Jpn. **66**, 3189 (1993).

[9] In elementary chaos, the equations of van del Pol and Duffing are well known to give typical examples, see Refs. [1,5] and [12].

[10] H. Hayashi, S. Ishizuka, M. Ohta, and K. Hirakawa, Phys. Lett. A **88**, 435 (1982).

[11] L. Glass, X. Guevara, and A. Shrier, Physica D **7**, 89 (1983).

[12] F. C. Moon, *Chaotic and Fractal Dynamics* (Wiley, New York, 1992).

[13] J. A. Yorke and E. Yorke, J. Statist. Phys. **21**, 263 (1979).

[14] A. Lasota and C. Mackey, J. Math. Biol. **10**, 333 (1980).

[15] S. J. Shenker and L. P. Kadanoff, J. Phys. A **14**, L23 (1981).

[16] H. Fujisaka, H. Kamifukamoto, and M. Inoue. Prog. Theoret. Phys. **69**, 333 (1983).

[17] C. Grebogi, E. Ott, and J. Yorke, Phys. Rev. Lett. **57**, 1284 (1986).

[18] O. Sporns, S. Roth, and F. F. Seelig, Physica D **26**, 215 (1987).

[19] R. M. Everson, Phys. Lett. A **122**, 471 (1987).

[20] M. C. Mackey and J. G. Milton, J. Math. Biol. **28**, 33 (1990).

[21] R. M. May, Nature **261**, 459 (1976).

[22] C. Grebogi, E. Ott, and J. Yorke, Phys. Rev. Lett. **48**, 1507 (1982); C. Grebogi, E. Ott, and J. A. Yorke, Physica D **7**, 181 (1983); C. Grebogi, E. Ott, F. Romeiras, and J. A. Yorke, Phys. Rev. A **36**, 5365 (1987).

[23] R. H. Abraham and H. B. Stewart, Physica D **21**, 394 (1986).

[24] Many examples of the crises and hierarchical crises can be seen in K. Takatsuka, Physica D **82**, 95 (1995).

[25] L. E. Reichl, *The Transition to Chaos* (Springer, Berlin, 1992).

[26] L. S. Liebovitch and T. I. Toth, J. Theoret. Biol. **148**, 243 (1991).

[27] L. S. Liebovitch and F. P. Czegledy, Ann. Biomed. Eng. **20**, 517 (1992).